Characterizations of several Maltsev Conditions

Marcin Kozik, Andrei Krokhin, Matt Valeriote, and Ross Willard

Abstract. Tame congruence theory identifies six Maltsev conditions associated with locally finite varieties omitting certain types of local behaviour. Extending a result of Siggers, we show that of these six Maltsev conditions only two of them are equivalent to strong Maltsev conditions for locally finite varieties. Besides omitting the unary type [24], the only other of these conditions that is strong is that of omitting the unary and affine types.

We also provide novel presentations of some of the above Maltsev conditions.

1. Introduction

A powerful tool for classifying varieties of algebras emerged in the 1960s, motivated by results of Maltsev [20] on congruence permutable varieties and by Jónsson [16] on congruence distributive varieties. They show that these congruence conditions on varieties can be expressed via what is now called a Maltsev condition. Since then, many important properties of varieties have been shown to be equivalent to Maltsev conditions. For background on the material discussed in this paper, the reader is directed to one of [7], [10], or [22].

Definition 1.1 ([13, 18])**.** (1) Let $\mathcal{U}$ and $\mathcal{V}$ be varieties and suppose that the operation symbols of $\mathcal{U}$ are $\{f_i : i \in I\}$. We say that $\mathcal{U}$ is *interpretable* in $\mathcal{V}$, and write $\mathcal{U} \leq \mathcal{V}$, if for every $i \in I$ there is a $\mathcal{V}$-term t_i of the same arity as f_i such that for all $\mathbf{A} \in \mathcal{V}$, the algebra $\langle A, t_i^{\mathbf{A}}(i \in I)\rangle$ is a member of $\mathcal{U}$.

(2) If $\mathcal{U}$ is a finitely presented variety, i.e., it has finitely many operation symbols and is finitely axiomatized, then the class of all varieties $\mathcal{V}$ with $\mathcal{U} \leq \mathcal{V}$ is called the *strong Maltsev class* defined by $\mathcal{U}$, and the condition $\mathcal{U} \leq \mathcal{V}$ on $\mathcal{V}$ is called the *strong Maltsev condition* defined by $\mathcal{U}$.

(3) If $\mathcal{U}_i$, $0 \leq i$, is a decreasing sequence of finitely presented varieties, relative to interpretability, then the class $\{\mathcal{V} : \mathcal{U}_i \leq \mathcal{V} \text{ for some } i\}$ is called the *Maltsev class* defined by this sequence, and the associated condition on varieties is called the *Maltsev condition* defined by this sequence.

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(4) An operation $f(\vec{x})$ on a set A is *idempotent* if the equation $f(x, x, \dots, x) \approx x$ holds. A term $t(\vec{x})$ of an algebra or variety is idempotent if the associated operation is, and we call an algebra or variety idempotent if all of its terms are.
(5) We say that a Maltsev condition is *proper* if it is not equivalent to a strong Maltsev condition. A (strong) Maltsev condition is idempotent if the (variety) varieties used to define it (is) are.

An idempotent Maltsev condition that we will study in this paper is that of being congruence n-permutable for some $n > 1$.

Definition 1.2. (1) The *relational product* of two binary relations R and S on a set A is the relation $R \circ S$ defined to be

$$\{(a, b) \in A^2 \,:\, (a, c) \in R \text{ and } (c, b) \in S \text{ for some } c \in A\}.$$

For $n > 1$, $R \circ_n S$ denotes the relation $R \circ S \circ R \cdots$ (with $n-1$ occurrences of $\circ$).
(2) For $n > 1$, an algebra $\mathbf{A}$ is *congruence n-permutable* if for all congruences α and β of $\mathbf{A}$, $\alpha \circ_n \beta = \beta \circ_n \alpha$. A variety is congruence n-permutable if all of its members are.

The following theorem, due to Hagemann and Mitschke, [14], shows that a variety being congruence n-permutable for some $n > 1$ is equivalent to an idempotent Maltsev condition and that for a fixed n, the class of congruence n-permutable varieties is a strong Maltsev class. In Section 3 we will show that the collection of all varieties that are congruence n-permutable for some n is a proper Maltsev class.

Theorem 1.3 ([14]). *Let $n > 0$. A variety $\mathcal{V}$ is congruence $(n+1)$-permutable if and only if it has ternary terms $p_1, \dots, p_n$ that satisfy the equations*

$$\begin{aligned} x &\approx p_1(x, y, y) \\ p_i(x, x, y) &\approx p_{i+1}(x, y, y) \quad \text{for each } i \\ p_n(x, x, y) &\approx y. \end{aligned}$$

In [15] a theory of the local structure of finite algebras is developed. The upshot of the theory is that there are exactly five types of local behaviour, numbered 1-5, that can be associated with locally finite varieties. For each locally finite variety $\mathcal{V}$ and each type $1 \le i \le 5$, $\mathcal{V}$ either *admits* i or *omits* i. The type will be often referred to by the following names:

- type **1** – the *unary type*,
- type **2** – the *affine type*,
- type **3** – the *Boolean type*,
- type **4** – the *lattice type*,
- type **5** – the *semilattice type*.

Type Omitting Class	Equivalent Property
$\mathcal{M}_{\{\mathbf{1}\}}$	satisfies a nontrivial idempotent Maltsev condition
$\mathcal{M}_{\{\mathbf{1},\mathbf{5}\}}$	satisfies a nontrivial congruence identity (see [18])
$\mathcal{M}_{\{\mathbf{1},\mathbf{4},\mathbf{5}\}}$	congruence n-permutable, for some $n > 1$
$\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$	congruence meet semidistributive
$\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{5}\}}$	congruence join semidistributive (see [18])
$\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}}$	congruence n-permutable for some n and congruence join semidistributive

Table 1. The six conditions

In Chapter 9 of [15] six type-omitting conditions are studied and, remarkably, are shown to be equivalent to idempotent Maltsev conditions, for locally finite varieties. There is a natural ordering on the five types of local behaviour, and these six conditions are associated with the downsets of types relative to this ordering. The classes of locally finite varieties corresponding to the six conditions are given in the following definition.

Definition 1.4. For I equal to one of the type sets $\{\mathbf{1}\}$, $\{\mathbf{1},\mathbf{2}\}$, $\{\mathbf{1},\mathbf{5}\}$, $\{\mathbf{1},\mathbf{2},\mathbf{5}\}$, $\{\mathbf{1},\mathbf{4},\mathbf{5}\}$, or $\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}$, let $\mathcal{M}_I$ denote the class of all locally finite varieties $\mathcal{V}$ that omit the types in the set I.

Hobby and Mckenzie show that these six classes can also be defined in terms of familiar conditions on congruence lattices. We note that in [15] none of the presentations of the Maltsev conditions for these six classes are strong. The six conditions are listed in Table 1. Some of these conditions have particularly nice descriptions in terms of interpretability and term conditions. The following definition and theorem expand on this.

Definition 1.5. Let $\mathbf{A}$ be an algebra, $\mathcal{V}$ a variety and $t(x_1,\dots x_n)$ a term for $\mathbf{A}$ or $\mathcal{V}$, for some $n > 0$.

(1) t is a *Taylor term* for $\mathbf{A}$ or $\mathcal{V}$ if it is idempotent and for each $1 \le i \le n$, an equation in the variables $\{x, y\}$ of the form $t(a_1,\dots,a_n) \approx t(b_1,\dots,b_n)$ holds, where $a_i \ne b_i$.

(2) t is a *Hobby-McKenzie* term for $\mathbf{A}$ or $\mathcal{V}$ if it is idempotent and for each non-empty $U \subseteq \{1,\dots,n\}$, an equation in the variables $\{x, y\}$ of the form $t(a_1,\dots,a_n) \approx t(b_1,\dots,b_n)$ holds, where $\{a_i : i \in U\} \ne \{b_i : i \in U\}$.

(3) t is called a *near unanimity term* for $\mathbf{A}$ or $\mathcal{V}$ if the equations

$$t(y,x,\dots,x) \approx t(x,y,x,\dots,x) \approx \cdots \approx t(x,x,\dots,x,y) \approx x$$

hold.

(4) t is called a *weak near unanimity term* for $\mathbf{A}$ or $\mathcal{V}$ if it is idempotent and the equations

$$t(y,x,\dots,x) \approx t(x,y,x,\dots,x) \approx \cdots \approx t(x,x,\dots,x,y)$$

hold.

Theorem 1.6. *Let $\mathcal{V}$ be a locally finite variety.*

(1) $\mathcal{V} \in \mathcal{M}_{\{\mathbf{1}\}}$ *if and only if for some $n > 1$, $\mathcal{V}$ has an n-ary Taylor term if and only if for some $n > 1$, $\mathcal{V}$ has an n-ary weak near unanimity term.*

(2) $\mathcal{V} \in \mathcal{M}_{\{\mathbf{1},\mathbf{5}\}}$ *if and only if for some $n > 1$, $\mathcal{V}$ has an n-ary Hobby-McKenzie term.*

(3) $\mathcal{V} \in \mathcal{M}_{\{\mathbf{1},\mathbf{4},\mathbf{5}\}}$ *if and only if for some $n > 1$, $\mathcal{V}$ is congruence n-permutable.*

(4) $\mathcal{V} \in \mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ *if and only if for all $n > 2$, $\mathcal{V}$ has an n-ary weak near unanimity term.*

Proof. The first statement follows from Lemma 9.4 and Theorem 9.6 of [15] and Theorem 1.1 of [21]. The second statement follows from Lemma 9.5 and Theorem 9.8 and the third from Theorem 9.14 of [15]. The last statement follows from Theorem 1.2 of [21] and the main result of [2] or [8]. □

A surprising result of Siggers [24], announced in 2008, and based on earlier work with Nešetřil [23], is that the class $\mathcal{M}_{\{\mathbf{1}\}}$ can in fact be defined by a strong Maltsev condition.

Theorem 1.7 ([24])**.** *Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{V}$ omits the unary type if and only if it has a 6-ary idempotent term t such that $\mathcal{V}$ satisfies the equations*

$$t(x,x,x,x,y,y) \approx t(x,y,x,y,x,x) \quad \textit{and} \quad t(y,y,x,x,x,x) \approx t(x,x,y,x,y,x).$$

One direction of this theorem follows by noting that any term t that satisfies the stated conditions is a Taylor term. In the next section we will present a proof of a variant of this theorem that uses a deep result of L. Barto and the first author. We will also establish a similar result for the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$.

2. Strong Maltsev Conditions

Shortly after the announcement of Siggers's result it was noted by Kearnes, Marković, and McKenzie, [17], that one could replace the 6-ary term of Siggers by one of several types of 4-ary terms. Their proof employs a deep result of Barto, Niven, and the first author, [5], on the complexity of the graph homomorphism problem. We make use of a different theorem of Barto and the first author on cyclic terms to establish one version of their result.

A term $t(x_1, \dots, x_n)$ of an algebra or variety is cyclic if it is idempotent and satisfies the equation $t(x_1, x_2, \dots, x_{n-1}, x_n) \approx t(x_2, x_3, \dots, x_n, x_1)$. Note that cyclic terms are special examples of weak near unanimity and Taylor terms.

Theorem 2.1 ([3])**.** *Let $\mathbf{A}$ be a finite algebra. Then $\mathsf{V}(\mathbf{A})$ omits the unary type if and only if for all prime numbers $p > |A|$, $\mathbf{A}$ has a p-ary cyclic term operation.*

Corollary 2.2 ([17]). *A locally finite variety $\mathcal{V}$ omits the unary type if and only if it has a 4-ary idempotent term operation t that satisfies the identities:*

$$t(x,y,z,y) \approx t(y,z,x,x).$$

Proof. It suffices to show that the free algebra $\mathbf{F}$ in $\mathcal{V}$ on 3 generators has such a term. Let p be some prime number larger than $|F|$ of the form $3k+2$ for some k and let $c(x_1,\dots,x_p)$ be a cyclic term of $\mathbf{F}$. Define $t(x,y,z,w)$ to be the term

$$c(x,x,\dots,x,y,y,\dots,y,w,z,z,\dots z),$$

where the variables x occurs $k+1$ times, the variable w occurs once, and the variables y and z occur k times each. Using that c is cyclic, it is easy to verify that $t(x,y,z,w)$ satisfies the stated equations in $\mathbf{F}$ and hence in $\mathcal{V}$.

Conversely, any term that satisfies the stated equations is a Taylor term and so any locally finite variety having such a term operation omits the unary type. □

Before considering the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$, we first show that one cannot expect a variety in $\mathcal{M}_{\{\mathbf{1}\}}$ to have a 3-ary Taylor term. The following is an example of a locally finite variety $\mathcal{V}$ which omits the unary type but has no Taylor term of arity less than 4, proving that the arity in Corollary 2.2 is optimal. This is also observed in [17].

Example 2.3. Let m and p be the unique majority and minority operations, respectively, on $\{0,1\}$, and let p_3 be the Maltsev operation on $\{0,1,2\}$ defined by $p_3(x,y,z) = x - y + z \pmod 3$. Let f be a ternary operation symbol, d a binary operation symbol, and define four finite algebras $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ as follows:

$$\begin{aligned}
A &= \{0,1\},\ f^{\mathbf{A}} = m,\ d^{\mathbf{A}}(x,y) = x\\
B &= \{0,1\},\ f^{\mathbf{B}} = p,\ d^{\mathbf{B}}(x,y) = x\\
C &= \{0,1,2\},\ f^{\mathbf{C}} = p_3,\ d^{\mathbf{C}}(x,y) = y\\
\mathbf{D} &= \mathbf{A}\times\mathbf{B}\times\mathbf{C}.
\end{aligned}$$

Note that the term $t(x,y,z,w) := d(f(x,y,z), f(y,w,z))$ satisfies the identities of Corollary 2.2 in each of $\mathbf{A},\mathbf{B},\mathbf{C}$ and hence also in $\mathbf{D}$, so $\mathbf{D}$ generates a variety which omits the unary type.

Theorem 2.4. $\mathbf{D}$ *has no Taylor term of arity less than 4.*

Proof. If it did, then it would have a Taylor term of arity 2 or 3. By padding with a dummy variable if necessary, $\mathbf{D}$ will have a ternary Taylor term, say $h(x,y,z)$. Let $h_1 = h^{\mathbf{A}}$, $h_2 = h^{\mathbf{B}}$, and $h_3 = h^{\mathbf{C}}$. Then h_1,h_2,h_3 are Taylor terms for $\mathbf{A},\mathbf{B},\mathbf{C}$ respectively.

The ternary terms of $\mathbf{A},\mathbf{B},\mathbf{C}$ are completely known:

- For $\mathbf{A}$ they are the projections and m. Of these, only m is a Taylor term for $\mathbf{A}$.

- For $\mathbf{B}$ they are the projections and p. Of these, only p is a Taylor term for $\mathbf{A}$.
- For $\mathbf{C}$ they are the projections, p_3, $q(x, y, z) := -(x + y) \pmod 3$, and the variants of p_3 and q obtained by permuting variables. Only the non-projections are Taylor terms for $\mathbf{C}$.

Thus $h_1 = m$, $h_2 = p$, and without loss of generality h_3 is either p_3 or q.

Case 1: $h_3 = p_3$.

We argue that there exists no choice of $a, b, c, d \in \{x, y\}$ such that

$$h(a, x, b) \approx h(c, y, d)$$

is an identity of $\mathbf{D}$. For if there were, then the same choice would produce identities

$$\begin{aligned} m(a,x,b) &\approx m(c,y,d) && \text{for } \mathbf{A} && (1)\\ p(a,x,b) &\approx p(c,y,d) && \text{for } \mathbf{B} && (2)\\ p_3(a,x,b) &\approx p_3(c,y,d) && \text{for } \mathbf{C}. && (3) \end{aligned}$$

If $a = x$, then identity (1) implies $c = d = x$. But neither $p_3(x, x, x) \approx p_3(x, y, x)$ nor $p_3(x, x, y) \approx p_3(x, y, x)$ is valid in $\mathbf{C}$, so $a = x$ is impossible. A similar argument rules out $b = x$ or $c = y$ or $d = y$. Thus we must have $a = b = y$ and $c = d = x$. But this contradicts identity (1), proving that a, b, c, d cannot be chosen as above, thus contradicting the fact that h is a Taylor term for $\mathbf{D}$.

Case 2: $h_3 = q$.

We argue that there exists no choice of $a, b, c, d \in \{x, y\}$ such that

$$h(a, b, x) \approx h(c, d, y)$$

is an identity of $\mathbf{D}$. For if there were, then the same choice would produce identities

$$\begin{aligned} m(a,b,x) &\approx m(c,d,y) && \text{for } \mathbf{A} && (4)\\ p(a,b,x) &\approx p(c,d,y) && \text{for } \mathbf{B} && (5)\\ -(a+b) &\approx -(c+d) && \text{for } \mathbf{C}. && (6) \end{aligned}$$

If $a = x$, then identity (4) implies $c = d = x$, which with identity (6) then implies $b = x$. But then identity (5) is not satisfied. A similar argument works if $b = x$ or $c = y$ or $d = y$. Thus we must have $a = b = y$ and $c = d = x$, which contradicts identity (4), proving that a, b, c, d cannot be chosen as above, thus contradicting the fact that h is a Taylor term for $\mathbf{D}$. □

In order to prove that the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ can be defined by a strong Maltsev condition we must first take a detour into ideas and results on the constraint satisfaction problem (CSP). For some background on the CSP, the reader is encouraged to consult [9] and more generally [11].

We present some standard definitions related to the CSP that have been suitably modified to meet our needs in this paper.

Definition 2.5. Let $\mathbf{A}$ be a finite algebra. An instance of the *constraint satisfaction problem* over $\mathbf{A}$ is a triple $P = (V, A, \mathcal{C})$ where

- V is a non-empty, finite set of variables and
- $\mathcal{C}$ is a set of constraints $\{C_1, \ldots, C_q\}$ where each C_i is a pair (S_i, R_i) with
  - S_i a non-empty subset of V called the scope of C_i, and
  - R_i is a subuniverse of the algebra $\mathbf{A}^{S_i}$, called the constraint relation of C_i.

We denote by $\mathsf{CSP}(\mathbf{A})$ the class of all instances of the CSP over $\mathbf{A}$.

A solution of P is a member $\vec{s}$ of A^V such that for all $i \leq q$, $\mathrm{restr}_{S_i}(\vec{s})$, the restriction of $\vec{s}$ onto the coordinates S_i, is a member of R_i.

Definition 2.6. Let $\mathbf{A}$ be a finite algebra and $P = (V, A, \mathcal{C}) \in \mathsf{CSP}(\mathbf{A})$.

(1) For $k > 0$, P is *k-minimal* if
  - For each subset I of V of size at most k, there is some constraint (S, R) in $\mathcal{C}$ such that $I \subseteq S$ and
  - If (S_1, R_1) and (S_2, R_2) are constraints in $\mathcal{C}$ and $I \subseteq S_1 \cap S_2$ has size at most k then $\mathrm{restr}_I(R_1) = \mathrm{restr}_I(R_2)$.

  For $I \subseteq V$ with $|I| \leq k$, these conditions allow us to define P_I to be the restriction of R onto I for some (or any) $(S, R) \in \mathcal{C}$ with $I \subseteq S$.

(2) P is *$(2,3)$-minimal* if it is 2-minimal and every three-element subset of V is included in the scope of some constraint

(3) $\mathbf{A}$ is said to be of *relational width k* (or width $(2,3)$) if every k-minimal ($(2,3)$-minimal) instance of $\mathsf{CSP}(\mathbf{A})$ whose constraint relations are all non-empty has a solution.

We invoke a key result from the theory of the CSP in order to prove the main result of this section.

Theorem 2.7 ([1, 4, 8]). *Let $\mathbf{A}$ be a finite idempotent algebra.*

(1) *(*[4, 8]*) $\mathbf{A}$ is of relational width k for some $k > 1$ if and only if $\mathsf{V}(\mathbf{A})$ omits the unary and affine types.*

(2) *(*[1]*) $\mathbf{A}$ is of width $(2,3)$ if and only if $\mathsf{V}(\mathbf{A})$ omits the unary and affine types.*

Theorem 2.8. *A locally finite variety omits the unary and affine types if and only if it has 3-ary and 4-ary weak near unanimity terms $v(x, y, z)$ and $w(x, y, z, w)$ that satisfy the equation $v(y, x, x) \approx w(y, x, x, x)$.*

Proof. We observe that no nontrivial variety of vector spaces can have a pair of weak near unanimity terms $v(x, y, z)$ and $w(x, y, z, w)$ that satisfy the equation $v(y, x, x) \approx w(y, x, x, x)$ and so by Theorem 9.10 of [15] we conclude that any locally finite variety that has such a pair of terms must omit the unary and affine types.

Conversely, if $\mathcal{V}$ omits the unary and affine types then we will build a $(2,3)$-minimal instance of the CSP over some finite algebra in $\mathcal{V}$ that, by Theorem 2.7, is guaranteed to have a solution. We may assume that $\mathcal{V}$ is idempotent since omitting the unary and affine types is determined by the idempotent terms of the variety.

Our construction is a variation of one found by E. Kiss to show that finite algebras of relational width k must have k-ary weak near unanimity terms. Let $\mathbf{F}$ be the $\mathcal{V}$-free algebra generated by $\{x, y\}$, let R be the subuniverse of $\mathbf{F}^3$ generated by

$$\{(y,x,x),(x,y,x),(x,x,y)\},$$

and S the subuniverse of $\mathbf{F}^4$ generated by

$$\{(y,x,x,x),(x,y,x,x),(x,x,y,x),(x,x,x,y)\}.$$

Let $n > (3|F|)$ and let $P = (V, F, \mathcal{C})$ be the following instance of the CSP:

- $V = \{x_1, \ldots, x_n\}$,
- $\mathcal{C}$ consists of constraints C_I for all $I \subseteq V$ with $|I| = 3$ or 4, where[1] $C_I = (I, R)$ if $|I| = 3$ and (I, S) if $|I| = 4$.

Since the subuniverses R and S are totally symmetric (i.e., closed under every permutation of their coordinates), and their restrictions onto any pair of coordinates are the same, it follows that P is a $(2,3)$-minimal instance over $\mathbf{F}$. Since $\mathbf{F}$ generates a variety that omits the unary and affine types then, by Theorem 2.7, we conclude that P has a solution $\vec{s} \in F^V$. Since $n > (3|F|)$ then by the Pigeon-Hole Principle it follows that $\vec{s}$ is constant on some $I \subseteq V$ with $|I| = 4$, say over the coordinates in I, $\vec{s}$ takes on the value $x \circ y \in F$, for some binary term $\circ$.

It follows that, since $\vec{s}$ is a solution of P and P contains the constraints (J, R) and (I, S) (where J is any 3 element subset of I), $(x \circ y, x \circ y, x \circ y) \in R$ and $(x \circ y, x \circ y, x \circ y, x \circ y) \in S$

Since R is generated by

$$\{(y,x,x),(x,y,x),(x,x,y)\},$$

and S by

$$\{(y,x,x,x),(x,y,x,x),(x,x,y,x),(x,x,x,y)\}$$

we conclude that there are terms $v(x,y,z)$ and $w(x,y,z,u)$ of $\mathcal{V}$ such that the following equations hold in $\mathbf{F}$:

$$v(y,x,x) \approx v(x,y,x) \approx v(x,x,y) \approx x \circ y$$

and

$$w(y,x,x,x) \approx w(x,y,x,x) \approx w(x,x,y,x) \approx w(x,x,x,y) \approx x \circ y.$$

Thus v and w are the desired terms of $\mathcal{V}$. $\square$

[1] More formally (for 3-element I) we choose any bijection, $f : \{1,2,3\} \to I$, and define the constraint relation to contain all g's such that $(g(f(1)), g(f(2)), g(f(3))) \in R$ and similarly for 4-element I.

If one would rather deal with k-minimality instead of $(2,3)$-minimality, then the above proof can easily be modified to show that algebras of relational width 3 must have 4-ary and 5-ary weak near unanimity terms r and s with $r(y,x,x,x) \approx s(y,x,x,x,x)$. This, of course, provides another strong Maltsev condition for omitting the unary and affine types.

Corollary 2.9. *The class $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ is defined by an idempotent strong Maltsev condition.*

Proof. Let $\mathcal{U}$ be the finitely presented variety with a 3-ary operation symbol v and a 4-ary operation symbol w defined by the equations that assert that v and w are weak near unanimity terms and that $v(y,x,x) \approx w(y,x,x,x)$. By the previous theorem we have that a locally finite variety $\mathcal{V}$ is in $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ if and only if $\mathcal{U} \leq \mathcal{V}$. □

3. Proper Maltsev Conditions

In this section we present a construction that we use to show that any strong Maltsev condition satisfied by all finitely generated varieties that are n-permutable for some $n > 1$ and that have a near unanimity term (and hence are congruence distributive and thus join semidistributive) is also satisfied by the variety of semilattices. Hence the "omitting types" classes $\mathcal{M}_{\{\mathbf{1},\mathbf{5}\}}$, $\mathcal{M}_{\{\mathbf{1},\mathbf{4},\mathbf{5}\}}$, $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{5}\}}$, and $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}}$ cannot be defined by strong Maltsev conditions, nor can the classes of locally finite varieties which are respectively congruence modular, congruence distributive, or that have a near unanimity term.

Put $\mathbf{2} = \{0,1\}$. For $N > 2$ and $1 \leq r \leq N-2$, define $\varphi_{N,r} : \mathbf{2}^N \to \mathbf{2}$ by

$$\varphi_{N,r}(\mathbf{x}) = \begin{cases} 1 & \text{if } |\{i : x_i = 1\}| > r \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\varphi_{N,r}$ is a near unanimity operation on $\mathbf{2}$. Also define the following ternary operations on $\mathbf{2}$:

$$\begin{aligned} \alpha_<(x,y,z) &= x \\ \alpha_=(x,y,z) &= x \vee (\overline{y} \wedge z) \\ \alpha_>(x,y,z) &= x \vee z. \end{aligned}$$

Definition 3.1. Fix $n \geq 2$. Define $N = n(n-1)^{n-1} + 1$ and, for $0 \leq i < n$, define $r_i = (n-1)^i$. Let $\mathcal{L}_n$ denote the language consisting of n ternary operation symbols $\mathsf{h}_0, \ldots, \mathsf{h}_{n-1}$ and one N-ary symbol q. For each $i < n$ define the algebra $\mathbf{D}[n,i]$ of type $\mathcal{L}_n$ by $\mathbf{D}[n,i] = (\mathbf{2}, \mathsf{h}_0^{\mathbf{D}[n,i]}, \ldots, \mathsf{h}_{n-1}^{\mathbf{D}[n,i]}, \mathsf{q}^{\mathbf{D}[n,i]})$ where

$$\begin{aligned} \mathsf{h}_j^{\mathbf{D}[n,i]} &= \begin{cases} \alpha_< & \text{if } j < i \\ \alpha_= & \text{if } j = i \\ \alpha_> & \text{if } j > i. \end{cases} \\ \mathsf{q}^{\mathbf{D}[n,i]} &= \varphi_{N,r_i}. \end{aligned}$$

Finally define $\mathbf{E}_n = \mathbf{D}[n,0] \times \mathbf{D}[n,1] \times \cdots \times \mathbf{D}[n,n-1]$ and put $\mathcal{V}_n = \mathrm{HSP}(\mathbf{E}_n)$.

Lemma 3.2. *For $n \geq 2$, $\mathcal{V}_n$ is $(2n+1)$-permutable and has an N-ary near unanimity term. $\mathcal{V}_n$ belongs to the class $\mathcal{M}_I$, for all six of the type sets I from Definition 1.4.*

Proof. It suffices to observe that the terms

$$\begin{aligned} p_{i+1}(x,y,z) &= \mathsf{h}_i(x,y,z) \quad \text{for } i<n \\ p_{2n-i}(x,y,z) &= \mathsf{h}_i(z,y,x) \quad \text{for } i<n \end{aligned}$$

satisfy the conditions from Theorem 1.3 and that $\mathsf{q}(x_1,\dots,x_N)$ is a near unanimity term for each $\mathbf{D}[n,i]$. □

We will eventually show that every at-most-n-ary term in $\mathcal{L}_n$ interprets in some $\mathbf{D}[n,i]$ as a join of variables. Before doing that, we prove some preliminary facts about operations on $\mathbf{2}$.

Fix $m \geq 1$. We let $\mathcal{F}_m$ denote the set of all functions $\mathbf{2}^m \to \mathbf{2}$, and let $\leq$ denote the usual (pointwise) order on $\mathcal{F}_m$. We write $\mathrm{pr}_1,\dots,\mathrm{pr}_m$ for the m-ary projection functions on $\mathbf{2}$. If $f \in \mathcal{F}_m$, then we say that f *dominates a projection* if there exists $1 \leq k \leq m$ such that $f \geq \mathrm{pr}_k$. For $S \subseteq \{1,\dots,m\}$ we let χ_S denote the element of $\mathbf{2}^m$ given by

$$\chi_S(k) = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.3. For $f, g \in \mathcal{F}_m$, define $f \rhd g$ iff for all $S \subseteq \{1,2,\dots,m\}$, if $g(\chi_S)=1$ then there exists $k \in S$ such that $f \geq \mathrm{pr}_k$.

Lemma 3.4.

(1) $\mathrm{pr}_k \rhd \mathrm{pr}_k$ *for all* $1 \leq k \leq m$.
(2) *If* $f_i \rhd g_i$ *for* $i=0,1$ *then* $(f_0 \vee f_1) \rhd (g_0 \vee g_1)$.
(3) *If* $f \rhd g$ *then* $f \geq g$.
(4) *If* $f_1 \geq f_0 \rhd g_0 \geq g_1$ *then* $f_1 \rhd g_1$.
(5) $f \in \mathcal{F}_m$ *is a term operation of* $(\mathbf{2},\vee)$ *iff* f *is idempotent and* $f \rhd f$.

Proof. We leave the proofs of the first four parts of this Lemma to the reader. For (5), suppose that $f \in \mathcal{F}_m$ is a term operation of $(\mathbf{2},\vee)$. Then $f = \bigvee_{j\in J} \mathrm{pr}_j$ for some subset J of $\{1,2,\dots,m\}$. Clearly f is idempotent. To show that $f \rhd f$, let $S \subseteq \{1,2,\dots,m\}$ with $f(\chi_S)=1$. This can only happen when $S \cap J \neq \emptyset$ and so there is some $k \in S \cap J$. But then $f \geq \mathrm{pr}_k$, as required.

Conversely, suppose that f is idempotent and $f \rhd f$. Let J be the set of all $j \in \{1,2,\dots,m\}$ with $f(\chi_{\{j\}})=1$ and let $g = \bigvee_{j\in J}\mathrm{pr}_j$. Since f is idempotent then $f(\chi_{\{1,2,\dots m\}})=1$ and so $f \rhd f$ implies that $f \geq \mathrm{pr}_i$ for some i. Thus J is nonempty and so g is well defined. We claim that $f=g$. That $f \geq g$ follows from the definition of J and $f \rhd f$. To establish equality, suppose that $S \subseteq \{1,2,\dots,m\}$ and $f(\chi_S)=1$. Then for some k, $f \geq \mathrm{pr}_k$, and from this we get that $k \in J$. It follows that $g(\chi_S)=1$ and so we conclude that $f=g$. □

Lemma 3.5. *Fix $n \geq 1$ and suppose $1 \leq r \leq N-2$ with $r < N/n$. Then for all $1 \leq m \leq n$, if $f_1, \dots, f_N \in \mathcal{F}_m$ are such that each f_i dominates a projection, then $\varphi_{N,r} \circ (f_1, \dots, f_N)$ also dominates a projection.*

Proof. Each f_i dominates at least one (out of m possible) projection, therefore there is a projection dominated by at least $\lceil N/m \rceil$ many f_i's. Since $r < N/n \leq \lceil N/m \rceil$ the function $\varphi_{N,r} \circ (f_1, \dots, f_N)$ dominates this projection as well. □

Lemma 3.6. *Fix $n \geq 2$ and suppose $1 \leq s \leq r \leq N-2$ with $r < N/n$ and $s \leq r/(n-1)$. Then for all $1 \leq m \leq n$, if $f_1, \dots, f_N, g_1, \dots, g_N \in \mathcal{F}_m$ are such that (i) each g_i dominates a projection, and (ii) $f_i \rhd g_i$ for all i, then*

$$\varphi_{N,s} \circ (f_1, \dots, f_N) \ \rhd \ \varphi_{N,r} \circ (g_1, \dots, g_N).$$

Proof. Let $\hat{f} = \varphi_{N,s} \circ (f_1, \dots, f_N)$, $\hat{g} = \varphi_{N,r} \circ (g_1, \dots, g_N)$ and let S be such that $\hat{g}(\chi_S) = 1$. By Lemma 3.5 $\hat{g}$ dominates a projection. Since $\hat{f} \geq \hat{g}$ the function $\hat{f}$ dominates the same projection and the case of $|S| = m$ is solved. From now on assume that $|S| \leq m-1$.

As $\hat{g}(\chi_S) = 1$ there are at least $(r+1)$-many g_i's such that $g_i(\chi_S) = 1$ and for each such g_i a corresponding f_i dominates a projection on a coordinate in S. Therefore we have at least $\lceil (r+1)/|S| \rceil$-many f_i's dominating a common projection in S. Since $s \leq r/(n-1) \leq r/|S| < \lceil (r+1)/|S| \rceil$ the function $\hat{f}$ dominates this projection as well. □

Lemma 3.7. *For any $n \geq 2$ and $1 \leq m \leq n$, if t is an m-ary term in $\mathcal{L}_n$ and $f_i = t^{\mathbf{D}[n,i]}$ for $i < n$, then:*

(1) *Each f_i is idempotent.*
(2) *f_{n-1} dominates a projection.*
(3) *$f_0 \rhd f_1 \rhd \cdots \rhd f_{n-1}$.*

Proof. The items are proved by induction on t. Note that all three items are clearly true if t is a variable (for item (3) use Lemma 3.4(1)), and that item (1) is easily seen to be true for all t. To prove (2) and (3), consider two cases.

In the first case $t = \mathsf{h}_j(r, s, u)$ (for some $j < n$) and r, s, u are m-ary terms for which the claims of the lemma are true. Let $r_i = r^{\mathbf{D}[n,i]}$, $s_i = s^{\mathbf{D}[n,i]}$ and $u_i = u^{\mathbf{D}[n,i]}$ for all i. Since $f_{n-1} \geq r_{n-1}$ and r_{n-1} dominates a projection, f_{n-1} dominates the same projection, proving (2). To prove (3) fix i such that $0 < i < n$. If $i < j$, then $(f_{i-1}, f_i) = (r_{i-1} \vee u_{i-1}, r_i \vee u_i)$ and so $f_{i-1} \rhd f_i$ using Lemma 3.4(2) and the fact that r, u satisfy item (3). If $i = j$ then $(f_{i-1}, f_i) = (r_{i-1} \vee u_{i-1}, r_i \vee (\overline{s_i} \vee u_i))$ and, by the same argument as above, $f_{i-1} \rhd r_i \vee u_i$. Finally $r_i \vee u_i \geq f_i$ by Lemma 3.4(4). If $i = j+1$ the reasoning is very similar. If $i > j+1$, then $(f_{i-1}, f_i) = (r_{i-1}, r_i)$ and so $f_{i-1} \rhd f_i$ since r satisfies item (3). Thus $f_{i-1} \rhd f_i$ for all $0 < i < n$, proving (3) in this case.

In the remaining case, assume that $u_1, \dots, u_N$ are m-ary terms in $\mathcal{L}_n$ for which the claims of the lemma are true, and $t = \mathsf{q}(u_1, \dots, u_N)$. Then we can deduce items (2) and (3) from Lemmas 3.5 and 3.6 respectively, carefully noting the interpretation of q in each $\mathbf{D}[n,i]$. □

We are now ready to prove the promised result about at-most-n-ary terms in $\mathcal{L}_n$.

Lemma 3.8. *Let $n \geq 2$ and suppose t is an m-ary term in $\mathcal{L}_n$. If $m \leq n$ then there exists $i < n$ such that $t^{\mathbf{D}[n,i]}$ is a term operation of $(\mathbf{2}, \vee)$.*

Proof. Let t be an m-ary term in $\mathcal{L}_n$ and define $f_i = t^{\mathbf{D}[n,i]}$ for $i < n$. We will assume that f_i is not a term operation of $(\mathbf{2}, \vee)$ for any $i < n$ and prove that $m > n$. For each $i < n$ define $T_i = \{k : \mathrm{pr}_k \leq f_i\}$ and note that $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_{n-1} \neq \varnothing$ by Lemmas 3.4(3) and 3.7(2,3).

For each $i < n$ we have $f_i \not\triangleright f_i$ by Lemma 3.4(5), so we may pick $S_i \subseteq \{1, 2, \ldots, m\}$ such that $f_i(\chi_{S_i}) = 1$ and yet $S_i \cap T_i = \varnothing$. We have that each $S_i \neq \varnothing$ because f_i is idempotent, and if $0 < i < n$ then $S_i \cap T_{i-1} \neq \varnothing$ because $f_{i-1} \triangleright f_i$. Hence $T_{i-1} \supsetneq T_i$ for all $0 < i < n$, proving $|T_0| \geq n$ and thus $m \geq |T_0 \cup S_0| = |T_0| + |S_0| \geq n + 1$. □

Theorem 3.9. *Any strong Maltsev condition satisfied by $\mathcal{V}_n$ for all $n \geq 2$ is also satisfied by the variety of semilattices.*

Proof. Since the variety of semilattices and $\mathcal{V}_n$ are idempotent for all $n \geq 2$ then we need only consider idempotent strong Maltsev conditions $\mathcal{U}$ in this proof. By making use of the idempotency of $\mathcal{U}$ we may assume that it can be presented in the form $\langle h(x_1, \ldots, x_m), \Sigma \rangle$ for some $m > 0$ and some finite set of equations Σ in the operation h. For example, if some idempotent finitely presented variety has operations $s(x_1, x_2, x_3)$ and $t(x_1, x_2)$ then an equivalent variety (with respect to interpretability) may be obtained by replacing s and t by a 6-ary operation $r(x_1, \ldots, x_6)$ and replacing all occurrences of $s(x_1, x_2, x_3)$ and $t(x_1, x_2)$ in the equations defining the variety by $r(x_1, x_1, x_2, x_2, x_3, x_3)$ and $r(x_1, x_2, x_1, x_2, x_1, x_2)$ respectively. In establishing one direction of this equivalence, r is set to be the term $s(t(x_1, x_2), t(x_3, x_4), t(x_5, x_6))$. Further details of this reduction can be found in the proof of Lemma 9.4 of [15] and also in [18].

To prove the theorem, it will suffice to show that if $\mathcal{U} \leq \mathcal{V}_n$ for some $n \geq m$ then $\mathcal{U} \leq$ *Semilattices*, the variety of semilattices. This follows from the previous Lemma, since if $n \geq m$ and the $\mathcal{V}_n$-term $t(x_1, \ldots, x_m)$ gives rise to an interpretation of $\mathcal{U}$ in $\mathcal{V}_n$ then for some $i < n$, $t^{\mathbf{D}[n,i]}$ is a term operation of $(\mathbf{2}, \vee)$. Thus the 2-element semilattice $(\mathbf{2}, \vee)$ has a term that also satisfies the equations in Σ and so the variety of semilattices interprets the variety $\mathcal{U}$. □

Corollary 3.10. *Of the six classes of locally finite varieties $\mathcal{M}_I$ from Definition 1.4, only $\mathcal{M}_{\{\mathbf{1}\}}$ and $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ can be defined by strong Maltsev conditions.*

Proof. That $\mathcal{M}_{\{\mathbf{1}\}}$ and $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ can be defined by strong Maltsev conditions is proved in Corollaries 2.2 and 2.9. Note that for all other type sets I from Definition 1.4, *Semilattices* $\notin \mathcal{M}_I$. If $\mathcal{U}$ is a strong Maltsev condition that is satisfied by all of the varieties in $\mathcal{M}_I$ then by Lemma 3.2 it is satisfied by $\mathcal{V}_n$ for all $n \geq 2$. Then by Theorem 3.9, the variety of semilattices also satisfies $\mathcal{U}$ and hence the strong Maltsev condition $\mathcal{U}$ cannot define the class $\mathcal{M}_I$. □

4. Matrix presentations of some Maltsev conditions

It has been noted, see for example [19], that term conditions similar to those listed in Definition 1.5 can sometimes be conveniently described using matrices over the variables x and y. If t is an n-ary term and $A = (a_{ij})$ is an $m \times n$ matrix of variables, then the expression $t[A]$ can be interpreted as the column vector whose ith entry is the term $t(a_{i1}, a_{i2}, \ldots, a_{in})$, for $1 \leq i \leq m$. Given two such expressions, $t[A]$, and $s[B]$, where the arities of t and s are n and k respectively, the expression $t[A] = s[B]$ can be interpreted as the system of m equations $t(a_{i1}, a_{i2}, \ldots, a_{in}) = s(b_{i1}, b_{i2}, \ldots, b_{ik})$, for $1 \leq i \leq m$.

Using this scheme, we see that an idempotent term $t(x_1, x_2, \ldots, x_n)$ will be a Taylor term for an algebra $\mathbf{C}$ (or variety $\mathcal{V}$) if and only if there are $n \times n$ matrices A and B over the variables x and y such that A has x's down its diagonal, B has y's down its diagonal and the system $t[A] = t[B]$ holds. In other words, t will be a Taylor term if a system of equations of the following form holds:

$$t \begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} \approx t \begin{bmatrix} y & & & \\ & y & & \\ & & \ddots & \\ & & & y \end{bmatrix},$$

where the non-diagonal entries of the two matrices can be filled in arbitrarily with x's and y's. The term t will be a near unanimity term if and only if it satisfies the equations given by the system $t[D] = X$ where D is the $n \times n$ matrix with y's down the diagonal and x's elsewhere and X is the $n \times 1$ matrix whose entries are all equal to x.

The following theorem provides matrix presentations for three of the classes found in Table 1. In the statement of the theorem, A and B are $n \times n$ matrices over the variables x and y and t is an n-ary idempotent term, for some $n > 0$.

Theorem 4.1. *([15], [6]) Let $\mathcal{V}$ be a locally finite variety.*

(1) *$\mathcal{V}$ is in $\mathcal{M}_{\{\mathbf{1}\}}$ if and only if it satisfies a system of the form $t[A] = t[B]$, where A has x's on its diagonal and B has y's on its diagonal.*
(2) *$\mathcal{V}$ is in $\mathcal{M}_{\{\mathbf{1},\mathbf{5}\}}$ if and only if it satisfies a system of the form $t[A] = t[B]$, where all of the entries of A on or below its diagonal are equal to x and the diagonal entries of B are all equal to y.*
(3) *$\mathcal{V}$ is in $\mathcal{M}_{\{\mathbf{1},\mathbf{2}\}}$ if and only if it satisfies a system of the form $t[A] = t[B]$, where A has x's on its diagonal and B has y's on its diagonal and all of the entries of A below its diagonal are equal to the corresponding entries of B below its diagonal.*

Proof. The first and second parts follow from Lemmas 9.4 and 9.5 and Theorems 9.6 and 9.8 of [15]. The third part can be found in [6]. □

In fact the first two matrix conditions from the theorem define, respectively, the class of all varieties (not necessarily locally finite) that satisfy a nontrivial idempotent Maltsev condition and the class of all varieties that satisfy an idempotent Maltsev condition that fails in the variety of semilattices (see Chapter 9 of [15]).

We next show that the class $\mathcal{M}_{\{\mathbf{1},\mathbf{4},\mathbf{5}\}}$, or more generally, the class of congruence n-permutable varieties for some $n > 1$, also has a matrix presentation. We wish to thank Benoit Larose for his contributions to the proof of this theorem.

Theorem 4.2. *Let $n > 0$ and $\mathcal{V}$ be a variety. If $\mathcal{V}$ is congruence $(n+1)$-permutable, then $\mathcal{V}$ has an idempotent term t of arity 3^n that satisfies a system of equations $t[A] = t[B]$ where A and B are $3^n \times 3^n$ matrices such that all of the entries of A on or below its diagonal are equal to x and all of the entries of B on or above its diagonal are equal to y. If $\mathcal{V}$ satisfies such a system for some m-ary idempotent term t, then $\mathcal{V}$ is congruence $(2m-1)$-permutable.*

Proof. Assume that $\mathcal{V}$ is congruence $(n+1)$-permutable and let $p_1, \dots, p_n$ be a sequence of terms for $\mathcal{V}$ that satisfies the equations from Theorem 1.3.

Let $3^n = \{0,1,2\}^n$ and $3^{\le n} = \bigcup_{k\le n} 3^k$. For $0 \le k \le n$ define a term $t_k(x_\sigma)_{\sigma\in 3^k}$ recursively as follows:

$$\begin{aligned} t_0(x_{\langle\rangle}) &= x_{\langle\rangle} \\ t_{k+1} &= p_{k+1}(t_k(x_{\sigma 0})_{\sigma\in 3^k}, t_k(x_{\sigma 1})_{\sigma\in 3^k}, t_k(x_{\sigma 2})_{\sigma\in 3^k}) \quad \text{for } 0 \le k < n. \end{aligned}$$

For each $k \le n$, let $\le$ denote the lexicographic ordering of 3^k. For example, for $k = 2$,

$$00 < 01 < 02 < 10 < 11 < 12 < 20 < 21 < 22.$$

We will construct suitable matrices A and B that witness, using the term t_n, the condition of the first part of this theorem.

Denote the least element of 3^k, i.e., $00\cdots 0$, by $\mathbf{0}$. For $k \le n$, define $\lambda_k, \mu_k : 3^k \to \{x,y\}$ so that for all $\sigma \in 3^k$:

$$\begin{aligned} \lambda_k(\sigma) = x \quad &\leftrightarrow \quad \sigma = \mathbf{0} \\ \mu_k(\sigma) = x \quad &\leftrightarrow \quad \sigma \in \{0,1\}^k. \end{aligned}$$

(In particular, $\lambda_0, \mu_0 : \langle\rangle \mapsto x$.) Also let $\hat{x}, \hat{y}$ denote the constant maps $3^k \to \{x,y\}$ whenever k is understood.

Claim: For all $0 \le k < n$, $V \models t_{k+1}(\lambda_{k+1}) \approx t_k(\mu_k)$.

Clearly $t_1(\lambda_1) = p_1(x,y,y) \approx x \approx t_0(x) = t_0(\mu_0)$. Inductively, for $k > 1$,

$$\begin{aligned} t_{k+1}(\lambda_{k+1}) &= p_{k+1}(t_k(\lambda_k), t_k(\hat{y}), t_k(\hat{y})) \\ &\approx p_{k+1}(t_{k-1}(\mu_{k-1}), y, y) \quad \text{(induction)} \\ &\approx p_k(t_{k-1}(\mu_{k-1}), t_{k-1}(\mu_{k-1}), y) \\ &\approx p_k(t_{k-1}(\mu_{k-1}), t_{k-1}(\mu_{k-1}), t_{k-1}(\hat{y})) \\ &= t_k(\mu_k). \end{aligned}$$

We now define the rows of the two $3^n \times 3^n$ matrices A and B needed for this direction of the theorem. The assignments defined in the following claim, with $k = n$, will form all but the first rows of A and B.

Claim: For all $1 \leq k \leq n$ and all $\sigma \in 3^k \setminus \{\mathbf{0}\}$, there exist assignments $f, g : 3^k \to \{x, y\}$ so that

(1) $f(\tau) = x$ for all $\tau \leq \sigma$,
(2) $g(\tau) = y$ for all $\tau \geq \sigma$, and
(3) $V \models t_k(f) \approx t_k(g)$.

For $k = 0$ the claim is vacuously true. Inductively assume that $k > 0$ and $\sigma \in 3^k \setminus \{\mathbf{0}\}$. Write $\sigma = \sigma_0 c$ with $\sigma_0 \in 3^{k-1}$ and $c \in \{0, 1, 2\}$.

Case 1: $\sigma_0 \neq \mathbf{0}$ (note that $k \neq 1$).

Then inductively there exist $f_0, g_0 : 3^{k-1} \to \{x, y\}$ such that $f_0(\tau_0) = x$ for all $\tau_0 \leq \sigma_0$, $g_0(\tau_0) = y$ for all $\tau_0 \geq \sigma_0$, and $V \models t_{k-1}(f_0) \approx t_{k-1}(g_0)$. Now define $f, g : 3^k \to \{x, y\}$ by putting $f(\tau_0 d) = f_0(\tau_0)$ and similarly for g. One can check that f and g satisfy (1), (2) and (3) in the statement of the claim.

Case 2: $\sigma_0 = \mathbf{0}$.

Define $f : 3^k \to \{x, y\}$ so that for all $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in 3^k$,

$$f(\sigma) = x \quad \leftrightarrow \quad \sigma_i \in \{0, 1\} \text{ for all } 1 \leq i < k.$$

Note that $f(\mathbf{0}d) = x$ for all $d \in \{0, 1, 2\}$. Furthermore,

$$\begin{aligned} t_k(f) &= p_k(t_{k-1}(\mu_{k-1}), t_{k-1}(\mu_{k-1}), t_{k-1}(\mu_{k-1})) \\ &\approx t_{k-1}(\mu_{k-1}) \\ &\approx t_k(\lambda_k) \quad \text{by the previous claim.} \end{aligned}$$

Thus f and g, with $g = \lambda_k$ satisfy the claim in this case.

To complete the proof of the first part of this theorem, we need to define the first rows of the two matrices. We note that the condition on the matrices forces the first row of B to consist entirely of y's and that the first entry of the first row of A to be equal to x. Setting the first row of A to be the assignment $h : 3^n \to \{x, y\}$ defined by $h(\sigma_1 \sigma_2 \cdots \sigma_n) = y$ iff $\sigma_n = 2$, we see that

$$\begin{aligned} t_n(h) &= p_n(t_{n-1}(\hat{x}), t_{n-1}(\hat{x}), t_{n-1}(\hat{y})) \\ &\approx p_n(x, x, y) \\ &\approx y \\ &\approx t_n(y, y, \ldots, y). \end{aligned}$$

Thus $\mathcal{V}$ satisfies the system $t_n[A] = t_n[B]$ and A and B have the required form.

For the second part of this theorem, suppose that $\mathcal{V}$ satisfies the system $t[A] = t[B]$ for some idempotent m-ary term t and $m \times m$ matrices A and B of the stated form. To show that $\mathcal{V}$ is congruence $(2m-1)$-permutable, we will define a sequence of ternary terms $p_i(x, y, z)$, $1 \leq i < 2m - 1$ that satisfy the equations in Theorem 1.3. For $1 \leq i \leq m$, define $f_i, g_i : \{1, 2, \dots, m\} \to \{x, y, z\}$ by

$$f_i(j) = \begin{cases} x & \text{if } j \leq i \\ y & \text{if } j > i \text{ and } A_{ij} = x \\ z & \text{otherwise} \end{cases}$$

and

$$g_i(j) = \begin{cases} B_{ij} & \text{if } j < i \\ z & \text{otherwise} \end{cases}$$

Let $p_1(x, y, z) = t(g_m)$ and for $1 \leq i < m$, let

$$p_{2i}(x, y, z) = t(f_{m-i}) \quad \text{and} \quad p_{2i+1}(x, y, z) = t(g_{m-i}).$$

Using the system of equations $t[A] = t[B]$ it is a routine exercise to show that the sequence of terms $p_1, \dots, p_{2m-2}$ satisfies the equations of Theorem 1.3. □

We next consider the two remaining classes from Table 1. The following theorem provides three equivalent conditions for membership in the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{5}\}}$. A variety is congruence join semidistributive (or satisfies $\mathsf{SD}(\vee)$) if for all $\mathbf{A} \in \mathcal{V}$ and all $\alpha, \beta, \gamma \in \mathrm{Con}\,(\mathbf{A})$, if $\alpha \vee \beta = \alpha \vee \gamma$ then $\alpha \vee \beta = \alpha \vee (\beta \wedge \gamma)$.

Theorem 4.3 ([15, 18])**.** *Let $\mathcal{V}$ be a variety. The following are equivalent:*

(1) *$\mathcal{V}$ is congruence join semidistributive.*
(2) *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in any nontrivial variety of modules and in the variety of meet semilattices.*
(3) *For some $n > 0$, $\mathcal{V}$ has terms $p_i(x, y, z)$ for $0 \leq i \leq n$ which satisfy the identities:*

$$\begin{aligned} p_0(x, y, z) &\approx x \\ p_n(x, y, z) &\approx z \\ p_i(x, y, y) &\approx p_{i+1}(x, y, y) \text{ and } p_i(x, y, x) \approx p_{i+1}(x, y, x) \text{ for all } i \text{ even} \\ p_i(x, x, y) &\approx p_{i+1}(x, x, y) \text{ for all } i \text{ odd} \end{aligned}$$

If $\mathcal{V}$ is locally finite, then these conditions are equivalent to $\mathcal{V}$ omitting the unary, affine, and semilattice types.

Definition 4.4. An idempotent term $t(x_1, \dots, x_n)$ of a variety $\mathcal{V}$ is called an $\mathsf{SD}(\vee)$-*term* for $\mathcal{V}$ if there are two $n \times n$ matrices A and B over the variables x and y such that the entries of A and B below their diagonals are all equal to x, the diagonal entries of A are all equal to x and the diagonal entries of B are all equal to y.

We note that the $\mathsf{SD}(\vee)$-term condition is the conjunction of the conditions from Parts 2 and 3 of Theorem 4.1 and so it follows that any locally finite variety that has an $\mathsf{SD}(\vee)$-term must belong to the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{5}\}}$ and is congruence join semidistributive. In the general case, we have the following.

Theorem 4.5. *Let $\mathcal{V}$ be a variety that has an $\mathsf{SD}(\vee)$-term. Then*

(1) *$\mathcal{V}$ is congruence join semidistributive and*
(2) *if $\mathcal{V}$ is locally finite then it belongs to the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{5}\}}$.*

Proof. The second part of this theorem has already been noted. To establish the first part, we need only show, by Theorem 4.3, that no nontrivial variety of modules or the variety of semilattices can have an $\mathsf{SD}(\vee)$-term. Since an $\mathsf{SD}(\vee)$-term is also a Hobby-McKenzie term, then by Lemma 9.5 of [15], the variety of meet semilattices does not have such a term. Let $\mathcal{M}$ be a nontrivial variety of modules and, to obtain a contradiction, assume that $t(x_1, \ldots, x_n)$ is an $\mathsf{SD}(\vee)$-term of $\mathcal{M}$. We may assume, without loss of generality, that t depends on all of its variables in $\mathcal{M}$. It follows from the $\mathsf{SD}(\vee)$-condition that the equation $x \approx t(x, x, \ldots, x) \approx t(x, x, \ldots, x, y)$ holds in $\mathcal{M}$. Since t is assumed to depend on the variable x_n, this equation cannot hold in a nontrivial module and thus $\mathcal{M}$ cannot have an $\mathsf{SD}(\vee)$-term. □

We present a partial converse to Theorem 4.5 by establishing it for congruence distributive varieties and point the reader to [12] for a proof of the full converse. We make use of the following characterization of congruence distributive varieties due to Jónsson.

Theorem 4.6 ([16])**.** *A variety is congruence distributive if and only if for some $n > 0$ it has terms $p_i(x, y, z)$, $0 \leq i \leq n$ that satisfy the equations*

$$\begin{aligned} p_0(x,y,z) &\approx x \\ p_n(x,y,z) &\approx z \\ p_i(x,y,x) &\approx x \text{ for all } i \\ p_i(x,x,y) &\approx p_{i+1}(x,x,y) \text{ for all } i \text{ even} \\ p_i(x,y,y) &\approx p_{i+1}(x,y,y) \text{ for all } i \text{ odd} \end{aligned}$$

Theorem 4.7. *If $\mathcal{V}$ is a congruence distributive variety then it has an $\mathsf{SD}(\vee)$-term.*

Proof. Our proof closely parallels the proof of Theorem 4.2 and the definition of the $\mathsf{SD}(\vee)$-term that we will build is almost identical to the one from the proof of that theorem, using the Jónsson terms from the previous theorem in place of the ternary terms for $(n+1)$-permutability. Instead, for $0 \leq k < n$, we set

$$t_{k+1} = p_{k+1}(t_k(x_{\sigma 0})_{\sigma \in 3^k}, t_k(x_{\sigma 2})_{\sigma \in 3^k}, t_k(x_{\sigma 1})_{\sigma \in 3^k}).$$

We now set out to show that the 3^n-ary term t_n is an $\mathsf{SD}(\vee)$-term for $\mathcal{V}$. Actually, since the equation $p_n(x, y, z) = z$ holds in $\mathcal{V}$, the term t_{n-1} also will work.

For $k \leq n$, define $\lambda_k, \mu_k : 3^k \to \{x, y\}$ so that for all $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in 3^k$: $\lambda_k(\sigma) = y$ if and only if $\sigma_k = 1$ and $\mu_k(\sigma) = x$ if and only if $\sigma_k = 0$. The following claim has a straightforward proof, using the idempotency of the terms involved.

Claim: For all $0 \leq k < n$, $V \models t_k(\lambda_k) \approx p_k(x, x, y)$ and $t_k(\mu_k) \approx p_k(x, y, y)$.

We now define the rows of the two $3^n \times 3^n$ matrices A and B needed for this direction of the theorem. The assignments defined in the following claim, with $k = n$, will form all but the first rows of A and B.

Claim: For all $1 \leq k \leq n$ and all $\sigma \in 3^k \setminus \{\mathbf{0}\}$, there exist assignments $f, g : 3^k \to \{x, y\}$ so that

(1) $f(\tau) = x$ for all $\tau \leq \sigma$,
(2) $g(\tau) = x$ for all $\tau < \sigma$,
(3) $g(\sigma) = y$, and
(4) $V \models t_k(f) \approx t_k(g)$.

For $k = 0$ the claim is vacuously true. Inductively assume that $k > 0$ and $\sigma \in 3^k \setminus \{\mathbf{0}\}$. Write $\sigma = \sigma_0 c$ with $\sigma_0 \in 3^{k-1}$ and $c \in \{0, 1, 2\}$.

Case 1: $\sigma_0 \neq \mathbf{0}$.

Then inductively there exist $f_0, g_0 : 3^{k-1} \to \{x, y\}$ such that $f_0(\tau_0) = x$ for all $\tau_0 \leq \sigma_0$, $g_0(\tau_0) = x$ for all $\tau_0 < \sigma_0$, $g_0(\sigma_0) = y$, and $V \models t_{k-1}(f_0) \approx t_{k-1}(g_0)$. Now define $f, g : 3^k \to \{x, y\}$ by

$$f(\tau_0 d) = \begin{cases} f_0(\tau_0) & \text{if } d = c \\ x & \text{if } d \neq c \end{cases}$$

and similarly for g. One can check that f and g satisfy the claim.

Case 2: $\sigma_0 = \mathbf{0}$.

If $c = 2$ then using the equation $p_k(x, y, x) \approx x$ it follows that the assignments $f(\tau) = x$ for all τ, and $g(\tau) = x$ for all τ, except when $\tau = \sigma$, work. We get that $t_k(f) \approx x \approx t_k(g)$ in this case.

The remaining case is when $c = 1$, or $\sigma = 000\cdots01$, and our argument breaks into two further cases, depending on whether k is even or odd. For k odd, let $g = \lambda_k$ and define $f : 3^k \to \{x, y\}$ so that $f(\tau_1\tau_2\cdots\tau_k) = \lambda_{k-1}(\tau_1\tau_2\cdots\tau_{k-1})$. It follows, using the previous claim, that

$$\begin{aligned} t_k(f) &= p_k(t_{k-1}(\lambda_{k-1}), t_{k-1}(\lambda_{k-1}), t_{k-1}(\lambda_{k-1})) \\ &\approx p_{k-1}(x, x, y) \\ &\approx p_k(x, x, y) \approx t_k(g). \end{aligned}$$

Since $f(\mathbf{0}) = f(\sigma) = g(\mathbf{0}) = x$ and $g(\sigma) = y$, then f and g work in this case, when k is odd. The even case can be handled similarly, using μ_k in place of λ_k.

To complete the proof of this theorem, we need to define the first rows of the two matrices. Setting all entries of the first row of the matrix A to y, except in the first place, and all entries of the first row of B to y works, since the equation $t_n(x, y, y, \ldots, y) \approx y \approx t_n(y, y, \ldots, y)$ holds in $\mathcal{V}$.

Thus $\mathcal{V}$ satisfies the system $t_n[A] \approx t_n[B]$ and A and B have the required form. $\square$

The following theorem combines the $\mathsf{SD}(\vee)$ and n-permutability matrix term conditions to yield one that implies membership in the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}}$.

Theorem 4.8. *If a variety $\mathcal{V}$ satisfies a system of equations of the form $t[A] \approx t[B]$ for some idempotent n-ary term t and two $n\times n$-matrices A and B over the variables x and y such that the entries of A on or below its diagonal are equal to x and the entries of B below its diagonal are equal to x and are equal to y elsewhere, then $\mathcal{V}$ is congruence join semidistributive and congruence $(2n-1)$-permutable. If $\mathcal{V}$ is locally finite, then it belongs to the class $\mathcal{M}_{\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}}$.*

We conclude with the following conjecture.

Conjecture 4.9. If $\mathcal{V}$ is a variety that is congruence join semidistributive and is congruence m-permutable for some $m > 1$ then it satisfies a system of equations as described in Theorem 4.8.

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Marcin Kozik

Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
e-mail: `marcin.kozik@tcs.uj.edu.pl`

Andrei Krokhin

School of Engineering and Computing Sciences, Durham University, Durham, UK
e-mail: `andrei.krokhin@durham.ac.uk`

Matt Valeriote

Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario, Canada
e-mail: `matt@math.mcmaster.ca`

Ross Willard

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada
e-mail: `rdwillar@uwaterloo.ca`